

# Propagation property and its application to inverse scattering for the fractional power of negative Laplacian

Atsuhide ISHIDA

Department of Liberal Arts, Faculty of Engineering,  
Tokyo University of Science  
3-1 Niijuku, 6-chome, Katsushika-ku, Tokyo 125-8585, Japan  
E-mail: aishida@rs.tus.ac.jp

## Abstract

Enss (1983) proved a propagation estimate for the usual free Schrödinger operator that turned out to be very useful for inverse scattering by Enss-Weder (1995). Since then, this method has been called the Enss-Weder time-dependent method. We study the same type of propagation estimate for the fractional power of the negative Laplacian and, as with the Enss-Weder method, we try to apply our estimate to inverse scattering. We find that the high velocity limit of the scattering operator uniquely determines the short-range interactions.

*Keywords:* scattering theory, inverse problem, fractional Laplacian  
*MSC2010:* 81Q10, 81U05, 81U40

## 1 Introduction

For  $1/2 \leq \rho \leq 1$ , the fractional power of the negative Laplacian as the self-adjoint operator acting on  $L^2(\mathbb{R}^n)$  is defined by the Fourier multiplier with the symbol

$$\omega_\rho(\xi) = |\xi|^{2\rho}/(2\rho). \quad (1.1)$$

We denote this operator by

$$H_{0,\rho} = \omega_\rho(D_x), \quad (1.2)$$

where  $D_x = -i\nabla_x = -i(\partial_{x_1}, \dots, \partial_{x_n})$ . More specifically, we can represent  $H_{0,\rho}$  by the Fourier integral operator

$$(H_{0,\rho}\phi)(x) = (\mathcal{F}^*\omega_\rho(\xi)\mathcal{F}\phi)(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}\omega_\rho(\xi)\phi(y)dyd\xi/(2\pi)^n \quad (1.3)$$

for  $\phi \in \mathcal{D}(H_{0,\rho}) = H^{2\rho}(\mathbb{R}^n)$ , which is the Sobolev space of order  $2\rho$ . In particular, if  $\rho = 1$ , then  $H_{0,1}$  is the free Schrödinger operator  $\omega_1(D_x) = -\Delta_x/2 = -\sum_{j=1}^n \partial_{x_j}^2/2$ . If  $\rho = 1/2$ , then  $H_{0,1/2}$  is the massless relativistic Schrödinger operator  $\omega_{1/2}(D_x) = \sqrt{-\Delta_x}$ .

In Section 2, we prove the following Enss-type propagation estimate for  $e^{-itH_{0,\rho}}$ . Throughout the paper,  $F(\dots)$  is the usual characteristic function of the set  $\{\dots\}$ . We denote the smooth characteristic function  $\chi \in C^\infty(\mathbb{R}^n)$  by

$$\chi(x) = \begin{cases} 1 & |x| \geq 2 \\ 0 & |x| \leq 1. \end{cases} \quad (1.4)$$

**Theorem 1.1.** *Let  $f \in C_0^\infty(\mathbb{R}^n)$  with  $\text{supp } f \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq \eta\}$  for some given  $\eta > 0$ . Choose  $v \in \mathbb{R}^n$  such that  $|v| \gg 1$ . The following estimate holds for  $t \in \mathbb{R}$  and  $N \in \mathbb{N}$ :*

$$\left\| \chi \left( \frac{x - (\nabla_\xi \omega_\rho)(v)t}{|v|^{2\rho-1}|t|/4} \right) e^{-itH_{0,\rho}} f(D_x - v) F \left( |x| \leq \frac{|v|^{2\rho-1}|t|}{16} \right) \right\| \leq C_N (|v|^{2\rho-1}|t|)^{-N}, \quad (1.5)$$

where  $\|\cdot\|$  stands for the operator norm on  $L^2(\mathbb{R}^n)$ , and the constant  $C_N > 0$  also depends on the dimension  $n$  and the shape of  $f$ .

Enss [4] proved the following estimate for the free Schrödinger operator:

$$\left\| F \left( |x - vt| \geq \frac{|v||t|}{4} \right) e^{-itD_x^2/2} f(D_x - v) F \left( |x| \leq \frac{|v||t|}{16} \right) \right\| \leq C_N (1 + |v||t|)^{-N}. \quad (1.6)$$

This estimate was proved not only for spheres but more generally for measurable subsets of  $\mathbb{R}^n$  (see Proposition 2.10 in Enss [4]). Before considering Theorem 1.1 further, we discuss the meaning of the estimate (1.6). From the perspective of classical mechanics,  $D_x$  represents the momentum or, in particular, the velocity of the particle when the mass is equal to 1. On the left-hand side of (1.6),  $D_x$  is localized to the neighborhood of  $v$  by the cut-off function  $f$ . Therefore, along the time evolution of the propagator  $e^{-itD_x^2/2}$ , the position of the particle behaves according to

$$x \sim D_x t \sim vt. \quad (1.7)$$

Because the behavior of the sphere is the same, the center of the sphere moves toward  $vt$  from the origin:

$$\left\{x \in \mathbb{R}^n \mid |x| \leq \frac{|v||t|}{16}\right\} \sim \left\{x \in \mathbb{R}^n \mid |x - vt| \leq \frac{|v||t|}{16}\right\}. \quad (1.8)$$

We can understand the meaning of the estimate (1.6) from these observations. The behavior of the sphere (1.8) makes the characteristic functions on both sides of (1.6) disjoint. Thus, this gives rise to the decay associated with time and velocity. Theorem 1.1 is the fractional Laplacian version of (1.6). Noting that  $(\nabla_\xi \omega_\rho)(v) = |v|^{2\rho-2}v$ , the case where  $\rho = 1$  in (1.5) is essentially equivalent to (1.6). Conversely, when  $\rho = 1/2$  in (1.5), the decay on the right-hand side does not involve  $|v|$ . However, this does not conflict with the physical meaning. In the case where  $\rho = 1/2$ , the system is relativistic. In this system, the particle does not have a mass, and its velocity is the light velocity, which is normalized to 1. Therefore, the decay cannot include the velocity  $v$ .

Spectral analysis for the relativistic Schrödinger operator was initiated by Weder [19], following which Umeda [15, 16] studied the resolvent estimate and mapping properties associated with the Sobolev spaces. Wei [22] also studied the generalized eigenfunctions. Weder [20] also analyzed the spectral properties of the fractional Laplacian for the massive case, and Watanabe [18] investigated the Kato-smoothness. Gierke [6] investigated the scattering theory and proved the asymptotic completeness of the wave operators in the case of short-range perturbations. Recently, Kitada [10, 11] constructed the long-range theory.

In Section 3, we assume that the space dimension satisfies  $n \geq 2$ . As an application of Theorem 1.1, we consider a multidimensional inverse scattering. The high-velocity limit of the scattering operator uniquely determines the interaction potentials that satisfy the short-range condition below by using the Enss-Weder time-dependent method (Enss-Weder [5]).

**Assumption 1.2.**  $V \in C^1(\mathbb{R}^n)$  is real-valued and satisfies, for  $\gamma > 1$ ,

$$|\partial_x^\beta V(x)| \leq C_\beta \langle x \rangle^{-\gamma-|\beta|}, \quad |\beta| \leq 1, \quad (1.9)$$

where the bracket of  $x$  has the usual definition  $\langle x \rangle = \sqrt{1 + |x|^2}$ .

For the full Hamiltonian  $H_\rho = H_{0,\rho} + V$ , where  $V$  belongs to the class above, the existence of the wave operators

$$W_\rho^\pm = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH_\rho} e^{-itH_{0,\rho}} \quad (1.10)$$

and their asymptotic completeness have already been proved (see Kitada [10] for example). Thus, we can define the scattering operator  $S_\rho = S_\rho(V)$  by

$$S_\rho = (W_\rho^+)^* W_\rho^-. \quad (1.11)$$

Under these situations, the following uniqueness theorem can be proved.

**Theorem 1.3.** *Let  $V_1$  and  $V_2$  be interaction potentials that satisfy Assumption 1.2. If  $S_\rho(V_1) = S_\rho(V_2)$ , then  $V_1 = V_2$  holds for  $1/2 < \rho \leq 1$ .*

We note that  $\rho = 1/2$  is excluded in this theorem. As mentioned before, in the case where  $\rho = 1/2$ , the system is relativistic and the light velocity is always equal to 1, that is,  $|v| \equiv 1$ . The Enss-Weder time-dependent method is also called the high-velocity method. As indicated by this name, deriving the uniqueness of the interaction potentials requires the limit of  $|v|$ . Thus, this method does not combine well with relativistic phenomena (see also Jung [9]).

In Enss-Weder [5], it was demonstrated that the estimate (1.6) was very useful for inverse scattering and the Enss-Weder time-dependent method was developed. Since then, the uniqueness of the interaction potentials for various quantum systems has been studied by many authors (Weder [21], Jung [9], Nicoleau [12, 13, 14], Adachi-Machara [3], Adachi-Kamada-Kazuno-Toratani [1], Valencia-Weder [17], Adachi-Fujiwara-Ishida [2] and Ishida [8]). This paper is motivated by these results. In particular, Enss-Weder [5] first proved the uniqueness of the potentials in the case where  $\rho = 1$  by applying (1.6). On the other hand, Jung [9] treated the case of  $\rho = 1/2$  using a slightly different approach. Of course, we cannot consider the limit of the velocity in this case. However, roughly speaking, Jung [9] translated the high-velocity limit into a high energy-limit and, without using an estimate of the type (1.5), obtained the uniqueness of the potentials. Thus, Theorem 1.3 represents an interpolation between the results of Enss-Weder [5] and Jung [9].

## 2 Propagation Property

In this section, we prove Theorem 1.1. Regarding estimate (1.6), the idea of Enss [4] was very simple and understandable. The Galilean transformation in the direction of  $v$  enabled reduction to a static system, and iteration of integration by parts, by taking the points of stationary phase into account, led to (1.6). However, in our case, these ingredients do not work well because of the fractional power. Instead, our main strategy is the asymptotic expansion of the symbolic calculus of pseudo-differential theory.

*Proof of Theorem 1.1.* By using unitary translations, we have the following relations:

$$e^{iv \cdot x} D_x e^{-iv \cdot x} = D_x - v, \quad (2.1)$$

$$e^{it\omega_\rho(D_x+v)} x e^{-it\omega_\rho(D_x+v)} = x + (\nabla_\xi \omega_\rho)(D_x + v)t. \quad (2.2)$$

We can thus compute that

$$\begin{aligned}
& \chi \left( \frac{x - (\nabla_{\xi} \omega_{\rho})(v)t}{|v|^{2\rho-1}|t|/4} \right) e^{-itH_{0,\rho}} f(D_x - v) \\
&= e^{iv \cdot x} \chi \left( \frac{x - (\nabla_{\xi} \omega_{\rho})(v)t}{|v|^{2\rho-1}|t|/4} \right) e^{-it\omega_{\rho}(D_x+v)} f(D_x) e^{-iv \cdot x} \\
&= e^{iv \cdot x} e^{-it\omega_{\rho}(D_x+v)} \chi \left( \frac{x + (\nabla_{\xi} \omega_{\rho})(D_x+v)t - (\nabla_{\xi} \omega_{\rho})(v)t}{|v|^{2\rho-1}|t|/4} \right) f(D_x) e^{-iv \cdot x}. \quad (2.3)
\end{aligned}$$

The plan of our proof is as follows. The momentum  $D_x$  can move inside the compact region only because of the cut-off  $f$ . Therefore,  $(\nabla_{\xi} \omega_{\rho})(D_x + v)$  and  $(\nabla_{\xi} \omega_{\rho})(v)$  almost cancel when  $|v|$  is sufficiently large, and the function  $\chi$  behaves as though  $\chi(x/(|v|^{2\rho-1}|t|/4))$ . We now justify this plan. Because  $|\xi| \leq \eta$  on the support of  $f$ , we have

$$|\xi + v| \geq |v| - |\xi| \geq |v| - \eta > 0. \quad (2.4)$$

This inequality says that

$$\chi \left( \frac{x + (\nabla_{\xi} \omega_{\rho})(\xi + v)t - (\nabla_{\xi} \omega_{\rho})(v)t}{|v|^{2\rho-1}|t|/4} \right) f(\xi) \in C^{\infty}(\mathbb{R}_x^n \times \mathbb{R}_{\xi}^n). \quad (2.5)$$

At the same time,

$$|(\nabla_{\xi} \omega_{\rho})(\xi + v) - (\nabla_{\xi} \omega_{\rho})(v)| \leq \int_0^1 |(\nabla_{\xi}^2 \omega_{\rho})(v + \theta\xi)| d\theta |\xi| \leq C|v|^{2\rho-2} \quad (2.6)$$

holds for  $|\xi| \leq \eta$ , where  $\nabla_{\xi}^2 \omega_{\rho}$  denotes the Hessian matrix of  $\omega_{\rho}$ . On the supports of  $f$  and  $\chi$ , we thus obtain

$$\begin{aligned}
|x| &\geq |x + (\nabla_{\xi} \omega_{\rho})(\xi + v)t - (\nabla_{\xi} \omega_{\rho})(v)t| - |(\nabla_{\xi} \omega_{\rho})(\xi + v) - (\nabla_{\xi} \omega_{\rho})(v)||t| \\
&\geq |v|^{2\rho-1}|t|/4 - C|v|^{2\rho-2}|t| \geq |v|^{2\rho-1}|t|/8
\end{aligned} \quad (2.7)$$

for large  $|v|$ . This means that

$$\begin{aligned}
& \chi \left( \frac{x + (\nabla_{\xi} \omega_{\rho})(\xi + v)t - (\nabla_{\xi} \omega_{\rho})(v)t}{|v|^{2\rho-1}|t|/4} \right) f(\xi) \\
&= \chi \left( \frac{x + (\nabla_{\xi} \omega_{\rho})(\xi + v)t - (\nabla_{\xi} \omega_{\rho})(v)t}{|v|^{2\rho-1}|t|/4} \right) f(\xi) \chi \left( \frac{x}{|v|^{2\rho-1}|t|/16} \right) \quad (2.8)
\end{aligned}$$

because  $\chi(x/(|v|^{2\rho-1}|t|/16)) = 1$  by (2.7). However, in the pseudo-differential calculus, the product of the symbols is not equal to the symbol of the product.

The additional asymptotic error terms come in. Symbolically, (2.8) becomes

$$\sum_{|\beta| \leq N-1} \frac{1}{\beta!} \partial_\xi^\beta \left\{ \chi \left( \frac{x + (\nabla_\xi \omega_\rho)(\xi + v)t - (\nabla_\xi \omega_\rho)(v)t}{|v|^{2\rho-1}|t|/4} \right) f(\xi) \right\} \\ \times (-i\partial_x)^\beta \chi \left( \frac{x}{|v|^{2\rho-1}|t|/16} \right) + R_N(t, x, \xi) \quad (2.9)$$

for any  $N \in \mathbb{N}$ . Every term with  $|\beta| \leq N-1$  disappears due to another characteristic function,

$$\left\{ (-i\partial_x)^\beta \chi \left( \frac{x}{|v|^{2\rho-1}|t|/16} \right) \right\} F \left( |x| \leq \frac{|v|^{2\rho-1}|t|}{16} \right) = 0. \quad (2.10)$$

Although the term  $R_N$  has a complicated shape, we know that this term includes the  $N$ -th derivative of  $\chi(x/(|v|^{2\rho-1}|t|/16))$  at  $x$ . Thus, it can be estimated by

$$\|R_N(t, x, D_x)\| \leq C_N(|v|^{2\rho-1}|t|)^{-N}. \quad (2.11)$$

□

### 3 Uniqueness of Interactions

To apply the Enss-Weder time-dependent method, we have to assume that  $n \geq 2$  and that  $\rho > 1/2$  from now on. The following Radon transformation-type reconstruction formula allows the proof of Theorem 1.3 to be performed. We devote ourselves to proving Theorem 3.1 in this section. Contrary to Enss-Weder [5], the key calculation in our proof is the pseudo-differential asymptotic expansion as in Theorem 1.1.

**Theorem 3.1.** *Let  $v \in \mathbb{R}^n$  be given and let  $\hat{v} = v/|v|$ . Suppose that  $\eta > 0$ , and that  $\Phi_0, \Psi_0 \in L^2(\mathbb{R}^n)$  such that  $\mathcal{F}\Phi_0, \mathcal{F}\Psi_0 \in C_0^\infty(\mathbb{R}^n)$  with  $\text{supp } \mathcal{F}\Phi_0, \text{supp } \mathcal{F}\Psi_0 \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq \eta\}$ . Let  $\Phi_v = e^{iv \cdot x} \Phi_0, \Psi_v = e^{iv \cdot x} \Psi_0$ . Then*

$$|v|^{2\rho-1}(i(S_\rho - 1)\Phi_v, \Psi_v) = \int_{-\infty}^{\infty} (V(x + \hat{v}t)\Phi_0, \Psi_0)dt + O(|v|^{\max\{1-2\rho+\epsilon, -1/(2+\gamma)\}}) \quad (3.1)$$

holds as  $|v| \rightarrow \infty$  for any  $0 < \epsilon < 2\rho - 1$  and any  $V$  that satisfies Assumption 1.2, where  $(\cdot, \cdot)$  is the scalar product of  $L^2(\mathbb{R}^n)$ .

The exponent of the convergence error in (3.1) is

$$\max\{1-2\rho+\epsilon, -1/(2+\gamma)\} = \begin{cases} 1-2\rho+\epsilon & \text{if } \rho \leq (3+\gamma)/\{2(2+\gamma)\} \\ -1/(2+\gamma) & \text{if } \rho > (3+\gamma)/\{2(2+\gamma)\}, \end{cases} \quad (3.2)$$

because  $\epsilon > 0$  can be chosen arbitrarily. Indeed, if  $\rho > (3 + \gamma)/\{2(2 + \gamma)\}$ , then we write  $\rho = (3 + \gamma)/\{2(2 + \gamma)\} + \delta$  for some  $\delta > 0$  and so  $1 - 2\rho + \epsilon = -1/(2 + \gamma) - 2\delta + \epsilon < -1/(2 + \gamma)$  holds for  $0 < \epsilon < 2\delta$ . If  $\rho \leq (3 + \gamma)/\{2(2 + \gamma)\}$ , then it is clear that  $1 - 2\rho + \epsilon \geq -1/(2 + \gamma) + \epsilon > -1/(2 + \gamma)$ . We emphasize that the error exponent is  $-1/(2 + \gamma)$  when  $\rho = 1$ . The corresponding order obtained by Enss-Weder [5] is  $o(1 - \gamma)$  for  $1 < \gamma < 2$  (see Theorem 2.4 in Enss-Weder [5]). Note that  $1 - \gamma > -1/(2 + \gamma)$  is equivalent to  $\gamma < (\sqrt{13} - 1)/2$ . Therefore, in the case where  $1 < \gamma < (\sqrt{13} - 1)/2$ , our exponent  $-1/(2 + \gamma)$  is better than the correspondence obtained by Enss-Weder [5].

We first prepare the propagation estimate of the following integral form. In the proof of this proposition, we can see that Theorem 1.1 plays an important role. While  $\|\cdot\|$  indicates the norm in  $L^2(\mathbb{R}^n)$ , for simplicity we do not distinguish between the notations for the usual  $L^2(\mathbb{R}^n)$ -norm and its operator norm in this paper.

**Proposition 3.2.** *Let  $v$  and  $\Phi_v$  be as in Theorem 3.1. Then*

$$\int_{-\infty}^{\infty} \|V(x)e^{-itH_{0,\rho}}\Phi_v\|dt = O(|v|^{1-2\rho}) \quad (3.3)$$

*holds as  $|v| \rightarrow \infty$  for any  $V$  that satisfies Assumption 1.2.*

*Proof.* Divide the integral into the domains  $|t| \leq |v|^{-\sigma}$  and  $|t| \geq |v|^{-\sigma}$ , so that

$$\int_{-\infty}^{\infty} \|V(x)e^{-itH_{0,\rho}}\Phi_v\|dt = \left( \int_{|t| \leq |v|^{-\sigma}} + \int_{|t| \geq |v|^{-\sigma}} \right) \|V(x)e^{-itH_{0,\rho}}\Phi_v\|dt, \quad (3.4)$$

with  $\sigma > 0$ , independent of  $t$  and  $v$ , to be determined below. It is clear that

$$\int_{|t| \leq |v|^{-\sigma}} \|V(x)e^{-itH_{0,\rho}}\Phi_v\|dt \leq C|v|^{-\sigma}, \quad (3.5)$$

because  $\|V(x)e^{-itH_{0,\rho}}\Phi_v\|$  is bounded uniformly in  $t$  and  $v$ . We now consider the case where  $|t| \geq |v|^{-\sigma}$ . Choose  $f \in C_0^\infty(\mathbb{R}^n)$  such that  $\mathcal{F}\Phi_0 = f\mathcal{F}\Phi_0$  and  $\text{supp } f \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq \eta\}$ . Then the relation

$$\Phi_v = e^{iv \cdot x} \mathcal{F}^* f(\xi) \mathcal{F} \Phi_0 = e^{iv \cdot x} f(D_x) \Phi_0 = f(D_x - v) \Phi_0 \quad (3.6)$$

can be seen. We thus compute

$$\|V(x)e^{-itH_{0,\rho}}\Phi_v\| = \|V(x)e^{-itH_{0,\rho}}f(D_x - v)\Phi_v\| \leq I_1 + I_2, \quad (3.7)$$

where  $I_1$  and  $I_2$  are given by

$$I_1 = \left\| V(x) \left\{ 1 - \chi \left( \frac{x - (\nabla_\xi \omega_\rho)(v)t}{|v|^{2\rho-1}|t|/4} \right) \right\} e^{-itH_{0,\rho}} f(D_x - v) \Phi_v \right\|, \quad (3.8)$$

$$I_2 = \left\| V(x) \chi \left( \frac{x - (\nabla_\xi \omega_\rho)(v)t}{|v|^{2\rho-1}|t|/4} \right) e^{-itH_{0,\rho}} f(D_x - v) \Phi_v \right\|. \quad (3.9)$$

When  $|x - (\nabla_\xi \omega_\rho)(v)t| \leq |v|^{2\rho-1}|t|/2$  holds, we have

$$|x| \geq |(\nabla_\xi \omega_\rho)(v)t| - |x - (\nabla_\xi \omega_\rho)(v)t| \geq |v|^{2\rho-1}|t|/2. \quad (3.10)$$

By virtue of the decay condition on  $V$  in (1.9) and inequality (3.10),  $I_1$  can be estimated as follows:

$$\begin{aligned} \int_{|t| \geq |v|^{-\sigma}} I_1 dt &\leq C \int_{|t| \geq |v|^{-\sigma}} \langle |v|^{2\rho-1}|t| \rangle^{-\gamma} dt \\ &\leq C |v|^{-(2\rho-1)\gamma} \int_{|v|^{-\sigma}}^{\infty} t^{-\gamma} dt = O(|v|^{-(2\rho-1)\gamma+\sigma(\gamma-1)}), \end{aligned} \quad (3.11)$$

because  $\gamma > 1$ . We next estimate  $I_2$ . Inserting

$$F\left(|x| \leq \frac{|v|^{2\rho-1}|t|}{16}\right) + F\left(|x| \geq \frac{|v|^{2\rho-1}|t|}{16}\right) = 1 \quad (3.12)$$

between  $f(D_x - v)$  and  $\Phi_v$ ,  $I_2$  can be estimated so that  $I_2 \leq I_{2,1} + I_{2,2}$  where  $I_{2,1}$  and  $I_{2,2}$  are given by

$$I_{2,1} = C \left\| \chi\left(\frac{x - (\nabla_\xi \omega_\rho)(v)t}{|v|^{2\rho-1}|t|/4}\right) e^{-itH_{0,\rho}} f(D_x - v) F\left(|x| \leq \frac{|v|^{2\rho-1}|t|}{16}\right) \right\|, \quad (3.13)$$

$$I_{2,2} = C \left\| F\left(|x| \geq \frac{|v|^{2\rho-1}|t|}{16}\right) \Phi_0 \right\|. \quad (3.14)$$

We can use Theorem 1.1 for  $I_{2,1}$  and, as with (3.11),

$$\int_{|t| \geq |v|^{-\sigma}} I_{2,1} dt \leq C \int_{|t| \geq |v|^{-\sigma}} \langle |v|^{2\rho-1}|t| \rangle^{-N} dt = O(|v|^{-(2\rho-1)N+\sigma(N-1)}) \quad (3.15)$$

is obtained for any  $N \geq 2$ .  $I_{2,2}$  can be made to have the same estimate as (3.15). Indeed,  $I_{2,2}$  satisfies

$$I_{2,2} \leq C \left\| F\left(|x| \geq \frac{|v|^{2\rho-1}|t|}{16}\right) \langle x \rangle^{-N} \right\| \|\langle x \rangle^N \Phi_0\| \leq C \langle |v|^{2\rho-1}|t| \rangle^{-N}, \quad (3.16)$$

because  $\Phi_0 \in \mathcal{S}(\mathbb{R}^n)$  by assumption. Therefore, we obtain

$$\int_{|t| \geq |v|^{-\sigma}} I_{2,2} dt = O(|v|^{-(2\rho-1)N+\sigma(N-1)}). \quad (3.17)$$

From (3.5), (3.11), (3.15) and (3.17), it follows that

$$\begin{aligned} &\int_{-\infty}^{\infty} \|V(x) e^{-itH_{0,\rho}} \Phi_v\| dt \\ &= O(|v|^{-\sigma}) + O(|v|^{-(2\rho-1)\gamma+\sigma(\gamma-1)}) + O(|v|^{-(2\rho-1)N+\sigma(N-1)}). \end{aligned} \quad (3.18)$$



To find the best exponent, we solve the equations

$$-\sigma = -(2\rho - 1)\gamma + \sigma(\gamma - 1) = -(2\rho - 1)N + \sigma(N - 1), \quad (3.19)$$

from which  $\sigma = 2\rho - 1 > 0$ .  $\square$

**Corollary 3.3.** *Let  $v$  and  $\Phi_v$  be as in Theorem 3.1. Then*

$$\|(W_\rho^\pm - 1)e^{-itH_{0,\rho}}\Phi_v\| = O(|v|^{1-2\rho}) \quad (3.20)$$

*holds as  $|v| \rightarrow \infty$  uniformly for  $t \in \mathbb{R}$ .*

*Proof.* The difference between  $W_\rho^\pm$  and 1 can be represented by the following integral form:

$$\begin{aligned} (W_\rho^\pm - 1)e^{-itH_{0,\rho}} &= \int_0^{\pm\infty} \partial_\tau e^{i\tau H_\rho} e^{-i\tau H_{0,\rho}} d\tau e^{-itH_{0,\rho}} \\ &= i \int_0^{\pm\infty} e^{i\tau H_\rho} V(x) e^{-i(\tau+t)H_{0,\rho}} d\tau = i \int_t^{\pm\infty} e^{i(\tau'-t)H_\rho} V(x) e^{-i\tau' H_{0,\rho}} d\tau'. \end{aligned} \quad (3.21)$$

In the last equation, we have used the change of variables  $\tau' = \tau + t$ . By using Proposition 3.2, we have

$$\|(W_\rho^\pm - 1)e^{-itH_{0,\rho}}\Phi_v\| \leq \int_{-\infty}^{\infty} \|V(x)e^{-i\tau' H_{0,\rho}}\Phi_v\| d\tau' = O(|v|^{1-2\rho}). \quad (3.22)$$

$\square$

We are ready to prove the reconstruction theorem.

*Proof of Theorem 3.1.* As in the proof of Corollary 3.3, we denote the difference between  $W^+$  and  $W^-$  by the integral

$$W_\rho^+ - W_\rho^- = \int_{-\infty}^{\infty} \partial_t e^{itH_\rho} e^{-itH_{0,\rho}} dt = i \int_{-\infty}^{\infty} e^{itH_\rho} V(x) e^{-itH_{0,\rho}} dt. \quad (3.23)$$

We recall intertwining property  $e^{-itH_\rho} W_\rho^\pm = W_\rho^\pm e^{-itH_{0,\rho}}$ . We can thus compute

$$\begin{aligned} i(S_\rho - 1)\Phi_v &= i(W_\rho^+ - W_\rho^-)^* W_\rho^- \Phi_v \\ &= \int_{-\infty}^{\infty} e^{itH_{0,\rho}} V(x) e^{-itH_\rho} W_\rho^- \Phi_v dt = \int_{-\infty}^{\infty} e^{itH_{0,\rho}} V(x) W_\rho^- e^{-itH_{0,\rho}} \Phi_v dt \end{aligned} \quad (3.24)$$

and

$$\begin{aligned} |v|^{2\rho-1} (i(S_\rho - 1)\Phi_v, \Psi_v) &= |v|^{2\rho-1} \int_{-\infty}^{\infty} (V(x) W_\rho^- e^{-itH_{0,\rho}} \Phi_v, e^{-itH_{0,\rho}} \Psi_v) dt \\ &= |v|^{2\rho-1} \int_{-\infty}^{\infty} I_v(t) dt + R_v, \end{aligned} \quad (3.25)$$

where we defined  $I_v(t)$  and  $R_v$  in (3.25) by

$$I_v(t) = (V(x)e^{-itH_{0,\rho}}\Phi_v, e^{-itH_{0,\rho}}\Psi_v), \quad (3.26)$$

$$R_v = |v|^{2\rho-1} \int_{-\infty}^{\infty} ((W_\rho^- - 1)e^{-itH_{0,\rho}}\Phi_v, V(x)e^{-itH_{0,\rho}}\Psi_v) dt. \quad (3.27)$$

Proposition 3.2 and Corollary 3.3 immediately give

$$R_v = O(|v|^{1-2\rho}). \quad (3.28)$$

Thus far, the proof is roughly parallel to that in Enss-Weder [5]. However, investigating the principal part of (3.25) demands another strict consideration. We first divide the integral as follows:

$$|v|^{2\rho-1} \int_{-\infty}^{\infty} I_v(t) dt = |v|^{2\rho-1} \left( \int_{|t| \leq |v|^{-\tilde{\sigma}}} + \int_{|t| \geq |v|^{-\tilde{\sigma}}} \right) I_v(t) dt, \quad (3.29)$$

where  $\tilde{\sigma} > 2\rho - 1$  is independent of  $t$  and  $v$ . We will determine an upper bound on  $\tilde{\sigma}$  later. Because  $I_v(t)$  is uniformly bounded in  $t$  and  $v$ , the integral on  $|t| \leq |v|^{-\tilde{\sigma}}$  is

$$|v|^{2\rho-1} \int_{|t| \leq |v|^{-\tilde{\sigma}}} I_v(t) dt \leq C|v|^{2\rho-1-\tilde{\sigma}}. \quad (3.30)$$

We next consider the integral on  $|t| \geq |v|^{-\tilde{\sigma}}$ , represented by

$$\begin{aligned} |v|^{2\rho-1} \int_{|t| \geq |v|^{-\tilde{\sigma}}} I_v(t) dt &= |v|^{2\rho-1} \int_{|t| \geq |v|^{-\tilde{\sigma}}} (V(x + (\nabla_\xi \omega_\rho)(v)t) \Phi_0, \Psi_0) dt \\ &\quad + |v|^{2\rho-1} \int_{|t| \geq |v|^{-\tilde{\sigma}}} \{ (V(x)e^{-itH_{0,\rho}}\Phi_v, e^{-itH_{0,\rho}}\Psi_v) \\ &\quad - (V(x + (\nabla_\xi \omega_\rho)(v)t) \Phi_0, \Psi_0) \} dt. \end{aligned} \quad (3.31)$$

We note that  $(\nabla_\xi \omega_\rho)(v) = |v|^{2\rho-2}v$ . By the change of variables  $\tau = |v|^{2\rho-1}t$ , the first term of (3.31) converges as  $|v| \rightarrow \infty$  because  $2\rho - 1 - \tilde{\sigma} < 0$ :

$$\begin{aligned} &|v|^{2\rho-1} \int_{|t| \geq |v|^{-\tilde{\sigma}}} (V(x + (\nabla_\xi \omega_\rho)(v)t) \Phi_0, \Psi_0) dt \\ &= \int_{|\tau| \geq |v|^{2\rho-1-\tilde{\sigma}}} (V(x + \hat{v}\tau) \Phi_0, \Psi_0) d\tau \longrightarrow \int_{-\infty}^{\infty} (V(x + \hat{v}\tau) \Phi_0, \Psi_0) d\tau. \end{aligned} \quad (3.32)$$

This also indicates that the error of the convergence in (3.32) is  $O(|v|^{2\rho-1-\tilde{\sigma}})$ . Recall the relations (2.1), (2.2) and the computation (2.3). We have

$$(V(x)e^{-itH_{0,\rho}}\Phi_v, e^{-itH_{0,\rho}}\Psi_v) = (V(x + (\nabla_\xi \omega_\rho)(D_x + v)t) \Phi_0, \Psi_0). \quad (3.33)$$

Therefore, as in the proof of Theorem 1.1, we try to derive the decay order in the second term of (3.31) from the near cancellation of  $(\nabla_\xi \omega_\rho)(D_x + v)$  and  $(\nabla_\xi \omega_\rho)(v)$  on the support of  $\Phi_0$ . By the asymptotic expansion of the pseudo-differential symbolic calculus, we obtain

$$\begin{aligned} & V\left(x + (\nabla_\xi \omega_\rho)(\xi + v)t\right) - V\left(x + (\nabla_\xi \omega_\rho)(v)t\right) \\ &= \int_0^1 (\nabla_x V)\left(x + (\nabla_\xi \omega_\rho)(v)t + \theta\{(\nabla_\xi \omega_\rho)(\xi + v) - (\nabla_\xi \omega_\rho)(v)\}t\right) \\ & \quad \cdot \{(\nabla_\xi \omega_\rho)(\xi + v) - (\nabla_\xi \omega_\rho)(v)\}t d\theta. \end{aligned} \quad (3.34)$$

We particularly note that the second- and higher-order derivatives of  $V$  do not appear on the right-hand side of (3.34) because  $(\nabla_\xi \omega_\rho)(\xi + v) - (\nabla_\xi \omega_\rho)(v)$  does not include  $x$ . Let  $f_1, f_2 \in C_0^\infty(\mathbb{R}^n)$  satisfy  $\mathcal{F}\Phi_0 = f_1 \mathcal{F}\Phi_0$  and  $f_1 = f_2 f_1$ . Then  $\Phi_0 = f_2(D_x) f_1(D_x) \Phi_0$  holds. We define  $g_{j,v}$  by

$$g_{j,v}(\xi) = \{(\partial_{\xi_j} \omega_\rho)(\xi + v) - (\partial_{\xi_j} \omega_\rho)(v)\} f_1(\xi) \quad (3.35)$$

for  $1 \leq j \leq n$  and, as in (2.6),

$$|\partial_\xi^\beta g_{j,v}(\xi)| \leq C_\beta |v|^{2\rho-2} \quad (3.36)$$

follows for any multi-index  $\beta$ . We also define the vector-valued function  $\psi_v$  by

$$\psi_v(t, x, \xi) = x + (\nabla_\xi \omega_\rho)(v)t + \theta\{(\nabla_\xi \omega_\rho)(\xi + v) - (\nabla_\xi \omega_\rho)(v)\}t \quad (3.37)$$

to avoid complicated notation. Now, consider the norm of the integrand of the  $j$ -th term on the right-hand side of (3.34):

$$|t| \|(\partial_{x_j} V)(\psi_v(t, x, D_x)) f_2(D_x) g_{j,v}(D_x) \Phi_0\| \leq J_1 + J_2, \quad (3.38)$$

where  $J_1$  and  $J_2$  are given by

$$\begin{aligned} J_1 &= |t| \left\| (\partial_{x_j} V)(\psi_v(t, x, D_x)) f_2(D_x) \chi\left(\frac{x}{|v|^{2\rho-1}|t|/4}\right) g_{j,v}(D_x) \Phi_0 \right\|, \\ J_2 &= |t| \left\| (\partial_{x_j} V)(\psi_v(t, x, D_x)) f_2(D_x) \left\{1 - \chi\left(\frac{x}{|v|^{2\rho-1}|t|/4}\right)\right\} g_{j,v}(D_x) \Phi_0 \right\|. \end{aligned} \quad (3.39)$$

For  $J_1$ , we insert

$$F\left(|x| \geq \frac{|v|^{2\rho-1}|t|}{4}\right) + F\left(|x| \leq \frac{|v|^{2\rho-1}|t|}{4}\right) = 1 \quad (3.40)$$

between  $g_{j,v}(D_x)$  and  $\Phi_0$ . Then  $J_1$  can be estimated so that  $J_1 \leq J_{1,1} + J_{1,2}$ , where  $J_{1,1}$  and  $J_{1,2}$  are given by

$$J_{1,1} = C|t| \left\| F \left( |x| \geq \frac{|v|^{2\rho-1}|t|}{4} \right) \Phi_0 \right\|, \quad (3.41)$$

$$J_{1,2} = C|t| \left\| \chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right) g_{j,v}(D_x) F \left( |x| \leq \frac{|v|^{2\rho-1}|t|}{4} \right) \Phi_0 \right\|. \quad (3.42)$$

The estimate of  $J_{1,1}$  is almost same as (3.16). However, in this estimate, we choose  $\nu \in \mathbb{R}$  instead of  $N \in \mathbb{N}$ :

$$J_{1,1} \leq C|t| \left\| F \left( |x| \geq \frac{|v|^{2\rho-1}|t|}{4} \right) \langle x \rangle^{-\nu} \right\| \|\langle x \rangle^\nu \Phi_0\| \leq C \langle |v|^{2\rho-1}|t| \rangle^{-\nu} |t|. \quad (3.43)$$

Therefore, for  $\nu > 2$ , we obtain

$$\begin{aligned} |v|^{2\rho-1} \int_{|t| \geq |v|^{-\tilde{\sigma}}} J_{1,1} dt &\leq C|v|^{2\rho-1} \int_{|t| \geq |v|^{-\tilde{\sigma}}} \langle |v|^{2\rho-1}|t| \rangle^{-\nu} |t| dt \\ &\leq C|v|^{-(2\rho-1)(\nu-1)} \int_{|v|^{-\tilde{\sigma}}}^{\infty} t^{-\nu+1} dt = O(|v|^{-(2\rho-1)(\nu-1)+\tilde{\sigma}(\nu-2)}). \end{aligned} \quad (3.44)$$

Although this estimate holds for any  $\nu > 2$ , the exponent is better when  $\nu$  is closer to 2 because

$$-(2\rho-1)(\nu-1) + \tilde{\sigma}(\nu-2) = (\nu-2)\{\tilde{\sigma} - (2\rho-1)\} + 1 - 2\rho \quad (3.45)$$

and  $\tilde{\sigma} > 2\rho-1$ . In the estimate of  $J_{1,2}$ , we compute the following commutator by using the pseudo-differential symbolic expansion:

$$\begin{aligned} &\left[ \chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right), g_{j,v}(\xi) \right] \\ &= - \sum_{1 \leq |\beta| \leq N-1} \frac{1}{\beta!} \partial_\xi^\beta g_{j,v}(\xi) \times (-i\partial_x)^\beta \chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right) + R'_N(t, x, \xi) \end{aligned} \quad (3.46)$$

for any  $N \in \mathbb{N}$ . As in the proof of Theorem 1.1, the disjointness of two characteristic functions means that, for  $0 \leq |\beta| \leq N-1$ ,

$$\left\{ (-i\partial_x)^\beta \chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right) \right\} F \left( |x| \leq \frac{|v|^{2\rho-1}|t|}{4} \right) = 0. \quad (3.47)$$

Therefore,  $J_{1,2}$  only has the remainder term  $R'_N$ . The remainder  $R'_N$  includes the  $N$ -th derivative of  $g_{j,v}$  and  $\chi$ . We thus estimate

$$J_{1,2} \leq C|t||v|^{2\rho-2}(|v|^{2\rho-1}|t|)^{-N}, \quad (3.48)$$

where we have used (3.36), and, for  $N \geq 3$ ,

$$\begin{aligned} |v|^{2\rho-1} \int_{|t| \geq |v|^{-\tilde{\sigma}}} J_{1,2} dt &\leq C |v|^{2(2\rho-1)-1-(2\rho-1)N} \int_{|v|^{-\tilde{\sigma}}}^{\infty} t^{-N+1} dt \\ &= O(|v|^{2(2\rho-1)-1-(2\rho-1)N+\tilde{\sigma}(N-2)}). \end{aligned} \quad (3.49)$$

This estimate holds for any  $N \geq 3$ . However, we can see that the best exponent is  $N = 3$  because

$$2(2\rho-1)-1-(2\rho-1)N+\tilde{\sigma}(N-2) = (N-2)\{\tilde{\sigma}-(2\rho-1)\}-1. \quad (3.50)$$

The right-hand side of (3.49) is then  $O(|v|^{\tilde{\sigma}-2\rho})$ . Finally, we consider  $J_2$ . On the supports of  $f_2$  and  $1-\chi$ ,

$$\begin{aligned} |\psi_v(t, x, \xi)| &\geq |v|^{2\rho-1}|t| - |x| - |(\nabla_{\xi}\omega_{\rho})(\xi+v) - (\nabla_{\xi}\omega_{\rho})(v)||t| \\ &\geq |v|^{2\rho-1}|t|/2 - C|v|^{2\rho-2}|t| \geq |v|^{2\rho-1}|t|/4 \end{aligned} \quad (3.51)$$

holds for large  $|v|$ , where we have used (2.6). This says that

$$\begin{aligned} f_2(\xi) &\left\{ 1 - \chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right) \right\} \\ &= \chi \left( \frac{\psi_v(t, x, \xi)}{|v|^{2\rho-1}|t|/8} \right) f_2(\xi) \left\{ 1 - \chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right) \right\}, \end{aligned} \quad (3.52)$$

because  $\chi(\psi_v(t, x, \xi)/(|v|^{2\rho-1}|t|/8)) = 1$  by (3.51). However, symbolically (3.51) is

$$\begin{aligned} \sum_{|\beta| \leq N-1} \frac{1}{\beta!} \partial_{\xi}^{\beta} \left\{ \chi \left( \frac{\psi_v(t, x, \xi)}{|v|^{2\rho-1}|t|/8} \right) f_2(\xi) \right\} \\ \times (-i\partial_x)^{\beta} \left\{ 1 - \chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right) \right\} + R_N''(t, x, \xi) \end{aligned} \quad (3.53)$$

for any  $N \in \mathbb{N}$  by using the asymptotic expansion again. For  $0 \leq |\beta| \leq N-1$ , we can use the decay condition on  $V$  in (1.9). We here note that

$$\left| \partial_{\xi}^{\beta} \chi \left( \frac{\psi_v(t, x, \xi)}{|v|^{2\rho-1}|t|/8} \right) \right| \quad (3.54)$$

can carry  $|v|^{1-2\rho}$ . However, it does not carry negative powers of  $|t|$  because of the definition of  $\psi_v$ . We now consider the integral over  $|t| \geq |v|^{-\tilde{\sigma}}$ , and a negative power of  $|t|$  has the harmful effect associated with the exponent of  $|v|$ . Therefore, for  $0 \leq |\beta| \leq N-1$ , we have to consider

$$\chi \left( \frac{\psi_v(t, x, \xi)}{|v|^{2\rho-1}|t|/8} \right) \partial_{\xi}^{\beta} f_2(\xi) \times \partial_x^{\beta} \left\{ 1 - \chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right) \right\} \quad (3.55)$$

only. The terms that include (3.55) are estimated so that

$$\begin{aligned}
& |t| \left\| (\partial_{x_j} V)(\psi_v(t, x, D_x)) \chi \left( \frac{\psi_v(t, x, D_x)}{|v|^{2\rho-1}|t|/8} \right) \right\| \\
& \quad \times \left\| \partial_x^\beta \left\{ 1 - \chi \left( \frac{x}{|v|^{2\rho-1}|t|/4} \right) \right\} \right\| \|g_{j,v}(D_x)\| \\
& \leq C|t| \langle |v|^{2\rho-1}|t| \rangle^{-1-\gamma} (|v|^{2\rho-1}|t|)^{-|\beta|} |v|^{2\rho-2}, \tag{3.56}
\end{aligned}$$

where we have used (1.9) and (3.36). We thus compute the following integral:

$$\begin{aligned}
& |v|^{2\rho-1} \int_{|t| \geq |v|^{-\tilde{\sigma}}} |t| \langle |v|^{2\rho-1}|t| \rangle^{-1-\gamma} (|v|^{2\rho-1}|t|)^{-|\beta|} |v|^{2\rho-2} dt \\
& \leq C|v|^{2(2\rho-1)-1-(2\rho-1)(1+\gamma)-(2\rho-1)|\beta|} \int_{|v|^{-\tilde{\sigma}}}^{\infty} t^{-\gamma-|\beta|} dt \\
& = O(|v|^{2(2\rho-1)-1-(2\rho-1)(1+\gamma)-(2\rho-1)|\beta|+\tilde{\sigma}(\gamma+|\beta|-1)}), \tag{3.57}
\end{aligned}$$

because  $\gamma > 1$ . The decay exponent in (3.57) can be represented by

$$\begin{aligned}
& 2(2\rho-1) - 1 - (2\rho-1)(1+\gamma) - (2\rho-1)|\beta| + \tilde{\sigma}(\gamma+|\beta|-1) \\
& = (\gamma+|\beta|-1)\{\tilde{\sigma} - (2\rho-1)\} - 1. \tag{3.58}
\end{aligned}$$

Because we assumed that  $\tilde{\sigma} > 2\rho-1$ , the top term between  $0 \leq |\beta| \leq N-1$  is  $|\beta| = N-1$ . The term involving  $R_N''$  includes the  $N$ -th derivative, and in the case where all derivatives at  $\xi$  operate  $f_2$  has the worst estimate. This is estimated by

$$|t| \|R_N''(t, x, D_x)\| \|g_{j,v}(D_x)\| \leq C|t| (|v|^{2\rho-1}|t|)^{-N} |v|^{2\rho-2}. \tag{3.59}$$

By the same argument as for the estimate of  $J_{1,2}$ , we fix  $N = 3$  and the integral of (3.59) is then

$$|v|^{2\rho-1} \int_{|t| \geq |v|^{-\tilde{\sigma}}} |t| (|v|^{2\rho-1}|t|)^{-3} |v|^{2\rho-2} dt = O(|v|^{\tilde{\sigma}-2\rho}). \tag{3.60}$$

From (3.57) and (3.60),  $J_2$  is estimated by

$$|v|^{2\rho-1} \int_{|t| \geq |v|^{-\tilde{\sigma}}} J_2 dt = O(|v|^{(\gamma+1)\{\tilde{\sigma}-(2\rho-1)\}-1}) + O(|v|^{\tilde{\sigma}-2\rho}). \tag{3.61}$$

By combining (3.28), (3.30), (3.32), (3.44), (3.49) and (3.61), we obtain

$$\begin{aligned}
|v|^{2\rho-1} (i(S_\rho - 1)\Phi_v, \Psi_v) &= \int_{-\infty}^{\infty} (V(x + \hat{v}t)\Phi_0, \Psi_0) dt \\
&+ O(|v|^{1-2\rho}) + O(|v|^{2\rho-1-\tilde{\sigma}}) + O(|v|^{\tilde{\sigma}-2\rho}) \\
&+ O(|v|^{(\gamma+1)\{\tilde{\sigma}-(2\rho-1)\}-1}) + O(|v|^{(\nu-2)\{\tilde{\sigma}-(2\rho-1)\}+1-2\rho}) \tag{3.62}
\end{aligned}$$

as  $|v| \rightarrow \infty$ . We next evaluate these error exponents. It is clear that  $1 - 2\rho < (\nu - 2)\{\tilde{\sigma} - (2\rho - 1)\} + 1 - 2\rho$ , and that we can choose  $\nu - 2 > 0$  to be sufficiently small, independent of the size of  $\tilde{\sigma}$ . Therefore, to determine  $\tilde{\sigma}$ , we should consider the relations between  $2\rho - 1 - \tilde{\sigma}$ ,  $\tilde{\sigma} - 2\rho$  and  $(\gamma + 1)\{\tilde{\sigma} - (2\rho - 1)\} - 1$ . These exponents must of course be negative. From  $(\gamma + 1)\{\tilde{\sigma} - (2\rho - 1)\} - 1 < 0$ , we obtain  $\tilde{\sigma} < 2\rho - 1 + 1/(1 + \gamma)$ . To show the convergence of Theorem 3.1 only, it is sufficient to choose  $\tilde{\sigma}$  such that

$$2\rho - 1 < \tilde{\sigma} < 2\rho - 1 + 1/(1 + \gamma), \quad (3.63)$$

because  $2\rho - 1 + 1/(1 + \gamma) < 2\rho$ . However, we want to find the best exponent. By solving

$$2\rho - 1 - \tilde{\sigma} = (\gamma + 1)\{\tilde{\sigma} - (2\rho - 1)\} - 1, \quad (3.64)$$

we obtain

$$\tilde{\sigma} = 2\rho - 1 + 1/(2 + \gamma) \quad (3.65)$$

and then both sides of (3.64) are equal to  $-1/(2 + \gamma)$ . Then  $(\nu - 2)\{\tilde{\sigma} - (2\rho - 1)\} + 1 - 2\rho = (\nu - 2)/(2 + \gamma) + 1 - 2\rho$  holds. For any  $0 < \epsilon < 2\rho - 1$ , we can choose  $\nu$  such that  $\epsilon = (\nu - 2)/(2 + \gamma)$ . This completes proof.  $\square$

From the Plancherel formula associated with the Radon transformation (see Helgason [7]), the proof of Theorem 1.3 can be performed in the same way as in Theorem 1.1 of Enss-Weder [5]. We thus omit the proof here.

**Acknowledgments.** This work was partially supported by the Grant-in-Aid for Young Scientists (B) #16K17633 from JSPS. Moreover, the author would like to thank the late Professor Hitoshi Kitada for many valuable discussions and comments.

## References

- [1] Adachi, T., Kamada, T., Kazuno, M., Toratani, K., On multidimensional inverse scattering in an external electric field asymptotically zero in time, *Inverse Problems* **27** (2011), no. 6, 065006. 17 pp.
- [2] Adachi, T., Fujiwara, Y., Ishida, A., On multidimensional inverse scattering in time-dependent electric fields, *Inverse Problems* **29** (2013), no. 8, 085012, 24 pp.
- [3] Adachi, T., Maehara, K., On multidimensional inverse scattering for Stark Hamiltonians, *J. Math. Phys.* **48** (2007), no. 4, 042101, 12 pp.

- [4] Enss, V., Propagation properties of quantum scattering states, *J. Funct. Anal.* **52** (1983), no. 2, 219–251.
- [5] Enss, V., Weder, R. A., The geometric approach to multidimensional inverse scattering, *J. Math. Phys.* **36** (1995), no. 8, 3902–3921.
- [6] Gierke, E., Asymptotic completeness for functions of the Laplacian perturbed by potentials and obstacles, *Math. Nachr.* **263/264** (2004), 133–153.
- [7] Helgason, S., *Groups and Geometric Analysis*, Academic Press, Orlando, 1984.
- [8] Ishida, A., On inverse scattering problem for the Schrödinger equation with repulsive potentials, *J. Math. Phys.* **55** (2014), no.8, 082101, 12 pp.
- [9] Jung, W., Geometrical approach to inverse scattering for the Dirac equation, *J. Math. Phys.* **38** (1997), no.1, 39–48.
- [10] Kitada, H., Scattering theory for the fractional power of negative Laplacian, *Jour. Abstr. Differ. Equ. Appl.* **1** (2010), no.1, 1–26.
- [11] Kitada, H., A remark on simple scattering theory, *Commun. Math. Anal.* **11** (2011), no.2, 123–138.
- [12] Nicoleau, F., Inverse scattering for Stark Hamiltonians with short-range potentials, *Asymptotic Anal.* **35** (2003), no. 3–4, 349–359.
- [13] Nicoleau, F., An inverse scattering problem for short-range systems in a time-periodic electric field, *Math. Res. Lett.* **12** (2005), no. 5–6, 885–896.
- [14] Nicoleau, F., Inverse scattering for a Schrödinger operator with a repulsive potential, *Acta Math. Sin. (Engl. Ser.)* **22** (2006), no. 5, 1485–1492.
- [15] Umeda, T., Radiation conditions and resolvent estimates for relativistic Schrödinger operators, *Ann. Inst. H. Poincaré Phys. Théor.* **63** (1995), no. 3, 277–296.
- [16] Umeda, T., The action of  $\sqrt{-\Delta}$  on weighted Sobolev spaces, *Lett. Math. Phys.* **54** (2000), no. 4, 301–313.
- [17] Valencia, G. D., Weder, R. A., High-velocity estimates and inverse scattering for quantum  $N$ -body systems with Stark effect, *J. Math. Phys.* **53** (2012), no. 10, 102105, 30 pp.
- [18] Watanabe, K., Smooth perturbations of the self-adjoint operator  $|\Delta|^{\alpha/2}$ , *Tokyo J. Math.* **14** (1991), no.1, 239–250.



- [19] Weder, R. A., Spectral properties of one-body relativistic Hamiltonians, *Ann. Inst. H. Poincaré Sect. A (N.S.)* **20** (1974), 211–220.
- [20] Weder, R. A., Spectral analysis of pseudodifferential operators, *J. Funct. Anal.* **20** (1975), no. 4, 319–337.
- [21] Weder, R. A., Multidimensional inverse scattering in an electric field, *J. Funct. Anal.* **139** (1996), no. 2, 441–465.
- [22] Wei, D., Completeness of eigenfunctions for relativistic Schrödinger operators I, *Osaka J. Math.* **44** (2007), no.4, 851–881.